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Ordinary differential equations which linearize on differentiation

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Abstract
In this paper, we discuss ordinary differential equations (ODEs) which linearize upon one (or more) differentiations. Although the subject is fairly elementary, equations of this type arise naturally in the context of integrable systems.

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1. Introduction

Let us consider a linear nth-order ODE with the general solution

$$u(x) = a_1 f_1(x) + \cdots + a_n f_n(x), \tag{1}$$

which is a linear superposition of n linearly independent solutions $f_i(x)$. Imposing a nonlinear relation among the coefficients,

$$F(a_1, \ldots, a_n) = 0,$$

one obtains an $(n-1)$-parameter family of functions $u(x)$ which automatically solve a nonlinear ODE of the order $n-1$. By construction, this ODE linearizes on differentiation. Imposing two relations among the coefficients, one obtains an ODE of the order $n-2$ which linearizes on two differentiations, etc. Using this simple recipe one can generate infinitely many examples of linearizable equations. This note was motivated by the observation that equations of this type arise *naturally* in the context of integrable systems. The paper is organized as follows.

Section 2 contains a list of examples of ODEs which linearize on differentiation. These equations appear in the construction of exact solutions of integrable PDEs, in the classification of integrable hydrodynamic chains, etc.

In section 3, we derive necessary and sufficient conditions for an ODE to linearize upon a finite number of differentiations.

In section 4, the general form of nonlinear ODEs linearizable by a differentiation is discussed. It is obtained by imposing nonlinear constraints among first integrals of a linear equation. Parallels with the theory of linear invariant subspaces of nonlinear differential operators are briefly discussed.

We point out that the problem of linearization of nonlinear ODEs has attracted a lot of attention in the literature. The conditions of point and contact linearizability of second- and third-order ODEs were first studied by Lie [13]; see also [1, p 38], [2, p 202], [3, 5, 11] and references therein. A short remark on 'integration via differentiation' can be found in Kamke [12, section 4.14]. We emphasize that in this paper we are concerned with a different concept of 'linearizability on differentiation'. Note that while the linearizability by point or contact transformations is closely connected with symmetry properties of the equation, this is not true in our case: the symmetry group of the original nonlinear equation can be trivial, becoming non-trivial for a linear equation obtained on differentiation.

2. Examples

Example 1. As shown in [10], the construction of 'follyton' solutions of a nonlinear system associated with a fourth-order self-adjoint spectral problem, reduces to an ODE

$$u''''u - u'''u' + \tfrac{1}{2}u''^2 = \tfrac{1}{2}c^4u^2,$$

$c = \text{const}$. On differentiation this equation becomes linear, $u''''' = c^4u'$, with the general solution

$$u = a_0 + a_1 \sinh cx + a_2 \cosh cx + a_3 \sin cx + a_4 \cos cx.$$

The substitution of this ansatz into the equation leads to a single quadratic relation among the coefficients, $-a_1^2 + a_2^2 + a_3^2 + a_4^2 = \frac{1}{2}a_0^2$.

Example 2. The classification of integrable Hamiltonian hydrodynamic chains associated with the Kupershmidt–Manin bracket reduces, in a particular case, to a solution of the nonlinear ODE [6]

$$\begin{aligned}(4c^2x^2u' - 12c^2xu - 1 - \alpha x + 2cx^2)u''' + (\alpha - 4cx + 12c^2u + 4c^2xu' - 2c^2x^2u'')u''\\ + (4c - 8c^2u')u' - \tfrac{1}{2} = 0,\end{aligned}$$

here c, α are arbitrary constants. Remarkably, this complicated equation linearizes on differentiation, taking the form

$$(4c^2x^2u' - 12c^2xu - 1 - \alpha x + 2cx^2)u'''' = 0.$$

Leaving aside the possibility that the coefficient at u'''' equals zero (see [6] for a complete analysis), we conclude that u must be a cubic polynomial,

$$u = a_0 + a_1x + a_2x^2 + a_4x^3,$$

where the constants satisfy a single relation $12a_4 - 8ca_1 + 16c^2\left(a_1^2 - 3a_2a_0\right) - 4a_2\alpha + 1 = 0$.

Example 3. Another subclass of integrable hydrodynamic chains from [6] is governed by the ODE

$$8x^2u'''u' + 8xu''u' - 4x^2u''^2 - u'^2 - 12u = 0$$

which linearizes on differentiation,

$$8x^2u'''' + 24xu''' + 6u'' - 12 = 0.$$

The general solution is given by the formula

$$u = x^2 + a_0 + a_1 x + a_2 x^{1/2} + a_3 x^{3/2}$$

where the constants satisfy a single quadratic relation $12a_0 + a_1^2 - 3a_2 a_3 = 0$.

Example 4. One of the versions of equations of associativity [4] reads as

$$F_{\xi\xi\xi} F_{\eta\eta\eta} - F_{\xi\xi\eta} F_{\xi\eta\eta} = 1.$$

Looking for solutions in the form $F = \xi^3 u(x)$, $x = \eta/\xi$, one arrives at the ODE

$$6uu''' - 4xu'u''' + 2xu''^2 - 2u'u'' = 1,$$

which takes the form $(6u - 4xu')u'''' = 0$ after a differentiation. The case $u'''' = 0$ leads to the general solution

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

where the constants satisfy a single quadratic relation $9a_0 a_3 - a_1 a_2 = 1$. In terms of F these solutions correspond to polynomials cubic in ξ and η. The case $6u - 4xu' = 0$ leads to $u = cx^{3/2}$. The corresponding F is given by the formula $F = c(\xi\eta)^{3/2}$ where $c = \mathrm{i}\frac{2\sqrt{2}}{3}$.

Example 5. The third-order ODE,

$$u''' = su'' \frac{(s+1)u' - 2xu''}{(s+1)((s+2)u - 2xu')}, \tag{2}$$

$s = \text{const}$ arises in the classification of integrable Hamiltonian hydrodynamic chains associated with Kupershmidt's brackets [7]. It possesses a remarkable property: for parameter values $s = 1, 2, 3, \ldots$ this equation linearizes on exactly s differentiations. Thus, for $s = 1$ the differentiation of (2) implies $u'''' = 0$, so that the general solution is

$$u = a_0 + 3a_1 x + 3a_2 x^2 + a_3 x^3,$$

where the constants a_i satisfy a single quadratic constraint $a_0 a_3 - a_1 a_2 = 0$. For $s = 2$, differentiating (2) twice, we arrive at $u^{(5)} = 0$ with the general solution

$$u = a_0 + 4a_1 x + 6a_2 x^2 + 4a_3 x^3 + a_4 x^4,$$

where the constants a_i satisfy a system of quadratic constraints

$$a_0 a_3 - a_1 a_2 = 0, \qquad a_1 a_4 - a_2 a_3 = 0, \qquad a_0 a_4 - a_2^2 = 0;$$

note that these constraints specify a determinantal variety characterized by the requirement that the rank of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & a_4 \end{pmatrix}$$

equals one. The mystery of this example is unveiled by the formula for its general solution,

$$u = a(x+c)^{s+2} + b(x-c)^{s+2},$$

which is valid for any s; here a, b, c are arbitrary constants (we thank A P Veselov for this observation).

3. Necessary and sufficient conditions for the linearizability

In this section, we demonstrate how to derive necessary and sufficient conditions for a nonlinear ODE to linearize on one (or more) differentiations. The procedure is fairly straightforward and can be readily adapted to particular situations.

3.1. First-order ODEs which linearize on one differentiation

Let us characterize first-order equations

$$u' = f(x, u)$$

which imply a linear equation,

$$u'' = a(x)u' + b(x)u + c(x),$$

on one differentiation. Thus, we have $f_x + f_u f = af + bu + c$. Differentiating this relation twice with respect to u, and introducing $F = f_x + f_u f$, one obtains $F_{uu} = af_{uu}$. Differentiating this by u once again one has $F_{uuu} = af_{uuu}$. Thus, the required linearizability condition takes the form

$$F_{uuu} f_{uu} = f_{uuu} F_{uu};$$

see section 4 for the general (implicit) form of all such right-hand sides $f(x, u)$ (formula (6)).

3.2. Second-order ODEs which linearize on one differentiation

Let us characterize second-order equations

$$u'' = f(x, u, p), \qquad p = u',$$

which imply a linear equation,

$$u''' = a(x)u'' + b(x)u' + c(x)u + k(x),$$

on one differentiation. Thus, we have $f_x + f_u p + f_p f = af + bp + cu + k$. Applying to this relation the operators $\partial_u^2, \partial_u\partial_p, \partial_p^2$, and introducing $F = f_x + f_u p + f_p f$, one obtains

$$F_{uu} = af_{uu}, \qquad F_{up} = af_{up}, \qquad F_{pp} = af_{pp},$$

or, equivalently, $\mathrm{d}^2 F = a\,\mathrm{d}^2 f$ (here the second symmetric differential d^2 is calculated with respect to u and p only). Differentiating this once again by u and p one obtains $\mathrm{d}^3 F = a\,\mathrm{d}^3 f$. Thus, the required linearizability condition takes the form

$$\mathrm{d}^3 F\,\mathrm{d}^2 f = \mathrm{d}^3 f\,\mathrm{d}^2 F.$$

3.3. First-order ODEs which linearize on two differentiations

Here we characterize first-order equations

$$u' = f(x, u)$$

which imply a linear equation

$$u''' = a(x)u'' + b(x)u' + c(x)u + k(x)$$

after two differentiations. Introducing $F = f_x + f_u f$ and $G = F_x + F_u f$, we have $G = aF + bf + cu + k$. Differentiating this twice with respect to u one obtains $G_{uu} = aF_{uu} + bf_{uu}$. This implies $G_{uuu} = aF_{uuu} + bf_{uuu}$ and $G_{uuuu} = aF_{uuuu} + bf_{uuuu}$. Thus, the required condition is

$$\det\begin{bmatrix} G_{uu} & F_{uu} & f_{uu} \\ G_{uuu} & F_{uuu} & f_{uuu} \\ G_{uuuu} & F_{uuuu} & f_{uuuu} \end{bmatrix} = 0.$$

In all of the above examples, the linearizability is characterized by differential relations which must be satisfied by the right-hand side of the equation. As we demonstrate in the next section, these differential equations can be integrated in closed form, leading to (implicit) representation for all linearizable equations.

4. General form of linearizable equations

All equations linearizable by a differentiation can be obtained by imposing functional relations among first integrals of linear equations. Since the first integrals can be parametrized explicitly by arbitrary functions of the independent variable x, this provides a general formula for equations which linearize on differentiation.

Example 6. Let us describe all ODEs which reduce to $u'''' = 0$ after one or two differentiations. The basis of first integrals consists of

$$\begin{aligned} I_1 &= u''', & I_2 &= xu''' - u'', \\ I_3 &= x^2u''' - 2xu'' + 2u', & I_4 &= x^3u''' - 3x^2u'' + 6xu' - 6u. \end{aligned} \tag{3}$$

Any third-order equation which reduces to $u'''' = 0$ after one differentiation can be represented by a single relation among the first integrals,

$$F(I_1, I_2, I_3, I_4) = 0.$$

Any second-order equation which reduces to $u'''' = 0$ after two differentiations can be represented in implicit form by two relations,

$$F(I_1, I_2, I_3, I_4) = 0, \qquad G(I_1, I_2, I_3, I_4) = 0;$$

one has to eliminate u''' to obtain the required second-order equation.

In general, let us consider a linear ODE

$$L[u] \equiv u^{(n)} + b_1(x)u^{(n-1)} + \cdots + b_{n-1}(x)u' + b_n(x)u = b(x). \tag{4}$$

Let $f_0(x)$ be its particular solution, and let $f_1(x), \ldots, f_n(x)$ be a fundamental system of solutions (FSS) of the corresponding homogeneous equation. A complete set of first integrals for the equation (4) can be taken in the form

$$I_i[u] = \frac{W[f_1, \ldots, f_{i-1}, u - f_0, f_{i+1}, \ldots, f_n]}{W[f_1, \ldots, f_n]}, \qquad i = 1, \ldots, n, \tag{5}$$

where $W[\cdot]$ denotes the Wronskian of the functions indicated in square brackets. Indeed, for an arbitrary solution $f(x) = f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x)$ we have $I_i[f(x)] = a_i$ for $i = 1, \ldots, n$, i.e., all $\{I_i\}$ take constant values. Applying (5) to the equation $u'''' = 0$ with $f_0(x) = 0$, $f_i(x) = x^{i-1}$, $i = 1, \ldots, 4$, one obtains first integrals which coincide with (3) up to constant factors.

For example, in the case $n = 2$ the general form of first-order equations linearizable by one differentiation is represented via arbitrary functions $f_0(x)$, $f_1(x)$, $f_2(x)$ and F as

$$F(I_1[u], I_2[u]) = 0 \tag{6}$$

where

$$I_1[u] = \frac{W[u - f_0, f_2]}{W[f_1, f_2]} = \frac{(u - f_0)f_2' - (u - f_0)'f_2}{f_1f_2' - f_1'f_2},$$

$$I_2[u] = \frac{W[f_1, u - f_0]}{W[f_1, f_2]} = \frac{-(u - f_0)f_1' + (u - f_0)'f_1}{f_1f_2' - f_1'f_2}.$$

The differentiation of (6) yields

$$\frac{-f_2F_{I_1} + f_1F_{I_2}}{(W[f_1, f_2])^2}((u - f_0)''W[f_1, f_2] - (u - f_0)'(W[f_1, f_2])' + (u - f_0)W[f_1', f_2']) = 0,$$

leading to the linear equation

$$W[f_1, f_2, u - f_0] = 0, \qquad \text{or} \qquad W[f_1, f_2, u] = W[f_1, f_2, f_0].$$

This construction generalizes to the case of arbitrary n in a straightforward way. In particular, equation (6) provides the general (implicit) form for equations discussed in subsection 3.1. Similar representations can be obtained for all other cases from section 3.

The linear span of the functions $f_i(x)$,

$$W_n = L\{f_1(x), \ldots, f_n(x)\}, \tag{7}$$

represents the linear space of solutions to the homogeneous equation $L[u] = 0$ corresponding to (4). The space W_n is said to be invariant with respect to a differential operator F, if $F[W_n] \subseteq W_n$. A systematic study of operators preserving a given subspace was initiated in [8] in the context of constructing explicit solutions for nonlinear evolution equations. The general form of operators preserving the subspace (7) is given by

$$F[u] = \sum_{i=1}^{n} A^i(I_1, \ldots, I_n) f_i(x), \tag{8}$$

where $A^i(I_1, \ldots, I_n)$ are arbitrary functions of the first integrals of the equation $L[u] = 0$ (see [9, 14] for more details). Given an operator F of the form (8), we introduce the equation

$$F[u] = 0, \tag{9}$$

and look for its solutions in the form (1). The substitution of (1) into (9) yields the identity

$$\sum_{i=1}^{n} A^i(I_1, \ldots, I_n) f_i(x) = 0$$

implying that

$$A^i(I_1, \ldots, I_n) = 0 \qquad \text{for} \quad i = 1, \ldots, n. \tag{10}$$

This system imposes relations on the coefficients a_i.

Most of the previous examples fit into this scheme. For instance, the equations from examples 2 and 4 are written via the first integrals (3) as

$$F[u] \equiv 2C^2\big(I_4 I_2 - I_3^2\big) + 2C I_3 - I_1 - \alpha I_2 - \tfrac{1}{2} = 0$$

and

$$F[u] \equiv -I_4 I_1 + I_3 I_2 - 1 = 0,$$

respectively, where I_i are given by (3). In both cases the system (10) reduces to a single relation.

Similarly, for the equation (2) rewritten as

$$F_s[u] \equiv (s+1)[(s+2)u'''u - su''u'] - 2x[(s+1)u'''u' - su''^2] = 0, \tag{11}$$

one obtains, for $s = 1$,

$$F_1[u] \equiv \tfrac{1}{2}(I_2 I_3 - I_1 I_4) = 0;$$

this again leads to a single relation (10). In the case $s = 2$ we have a representation

$$F_2[u] \equiv \tfrac{1}{2}\big[(I_2 I_3 - I_1 I_4)x^2 + \big(I_1 I_5 - I_3^2\big)x + (I_3 I_4 - I_2 I_5)\big] = 0 \tag{12}$$

via the first integrals $I_1, \ldots, I_5$ of the equation $u^{(5)} = 0$. In this case the system (10) formally consists of three relations,

$$I_2 I_3 - I_1 I_4 = 0, \qquad I_1 I_5 - I_3^2 = 0, \qquad I_3 I_4 - I_2 I_5 = 0,$$

however, only two of them are functionally independent.

Remark. In accordance with [15], every operator (8) of the order $n-1-k$, admitted by the equation

$$u^{(n)} = 0, \tag{13}$$

is expressed in terms of the differences

$$J_i^k = xJ_i^{k-1} - J_{i+1}^{k-1}, \qquad i = 1, \ldots, n-k, \tag{14}$$

with $J_i^0 \equiv I_i, i = 1, \ldots, n$ (I_i are first integrals for (13)). All these expressions are of the order $n-1-k$, and satisfy the identity $D^{k+1}J_i^k = 0$ on solutions of the equation (13). Setting $n = s+3$, $k = s-1$, we obtain that any third-order operator admitted by the equation $u^{(s+3)} = 0$ is defined via the functions $J_i^{s-1}, i = 1, \ldots, 4$. For instance, the operator F_s from (11) is represented as

$$F_s[u] = \tfrac{1}{2}\left(J_2^{s-1}J_3^{s-1} - J_1^{s-1}J_4^{s-1}\right).$$

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